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抽象度量空间上的一致连续函数空间

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抽象度量空间上的一致连续函数空间

BANACH SPACES OF UNIFORMLY CONTINUOUS FUNCTIONS ON ABSTRACT METRIC SPACES

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内 容 摘 要

本文研究抽象度量空间上的一致连续函数空间。本文分为两章。
在第一章中，我们主要研究一致连续函数空间的 Banach 空间性质，主要结论如下：

1．给出了标量值一致连续函数的线性，格以及代数性质；
2．证明了两种一致连续函数空间 $(UC(X), UC_0(X))$ 在各自适当的范数下是 Banach 空间，并且讨论了这两种一致连续函数空间和两种 Lipschitz 空间，有界连续函数空间及对偶空间的关系；
3．证明了两种由连续模确定的 Lipschitz 空间是对偶空间；
4．证明了一致连续函数空间 $UC_0(X)$ 的对偶空间的闭单位球的端点在一定条件下的存在性并且给出了端点的具体形式。

第二章主要研究一致连续函数空间的格结构和代数结构，主要结论如下：

1．证明了两种一致连续函数空间可以再赋范使之成为 Banach 代数；
2．给出了两种一致连续函数空间的 Stone-Weierstrass 定理；
3．证明了一致连续函数空间 $UC_0(X)$ 是一个 Lipschitz 格，从而 $UC_0(X)$ 格同构于某个 Lipschitz 空间 $Lip_0(Y)$。

关键词：Banach 空间，一致连续函数，Lipschitz 空间，Lipschitz 格

Abstract

This report is devoted to Banach spaces of uniformly continuous functions on abstract metric spaces. We have divided this report into two chapters.

In Chapter 1, we study Banach space properties of Banach spaces of uniformly continuous functions on abstract metric spaces and the main results are as follows:

1. **We give some elementary properties of scalar-valued uniformly continuous functions such as linear, lattice-theoretic and algebraic properties;**
2. **It is proved that two types of spaces of uniformly continuous functions** $(UC(X), UC_0(X))$ **are both Banach spaces under appropriate norms and then discuss the relationships between these two types of Banach spaces , two types of Lipschitz spaces, Banach spaces of scalar-valued bounded continuous functions and dual spaces;**
3. **Prove that two types of Banach spaces determined by a modulus of continuity are both dual spaces;**
4. **First ensure the existence of the extreme points of the unit ball of** $UC_0(X)^*$ **under some conditions and then give the exact form that the extreme points must take.**

Chapter 2 is devoted to lattice and algebraic structures of the two types of spaces of uniformly continuous functions. The main results are as follows:

1. **Show that** $UC(X), UC_0(X)$ **can be renormed to become Banach algebras;**
2. **Give the Stone-Weierstrass theorems for** $UC(X)$ **and** $UC_0(X)$**;**
3. **Prove that** $UC_0(X)$ **is a Lipschitz lattice and hence** $UC_0(X)$ **is lattice isomorphic to some** $Lip_0(Y)$**.**

Keywords: Banach spaces, uniformly continuous functions, Lipschitz spaces, Lipschitz lattices.

符 号 表

F:	标量域
R:	实数域
C:	复数域
ω_f ：	函数 f 的连续模
$UC(X)$ ：	X 上的有界的标量值一致连续函数空间
$UC_0(X)$：	X 上的保持基点的标量值一致连续函数空间
$Lip(X)$：	X 上的有界的标量值 Lipschitz 函数空间
$Lip_0(X)$：	X 上的保持基点的标量值 Lipschitz 函数空间
$C_b(X)$：	X 上的有界的标量值的连续函数空间
ρ：	度量
X^*：	X 的对偶空间
$l^\infty(X)$：	X 上的有界的标量值函数空间
$B(X)$：	X 的闭单位球
$ext(B(X))$：	X 的闭单位球的端点的全体
$C(X)$：	X 上的标量值的连续函数空间
$M(X)$：	X 上的正则 Borel 测度的全体
$\mathrm{Re}\, f$ ：	函数 f 的实部
$\mathrm{Im}\, f$ ：	函数 f 的虚部
N：	自然数的全体

目　次

Chapter 0

Introduction

Let X be a metric space and let $e \in X$ be a distinguished base point. Then the Banach space $Lip_0(X)$ is defined as the set of all scalar-valued Lipschitz functions on X which vanish at e,with the norm of a function f its Lipschitz number $L(f)$,

$$L(f) = \sup_{p \neq q} \frac{|f(p) - f(q)|}{\rho(p, q)}$$

The reason for requiring functions to vanish at a base point is to exclude constant functions, which have Lipschitz number 0;thus $L(\cdot)$ becomes a norm rather than merely a seminorm. Another way to obtain a norm is to consider all bounded scalar-valued Lipschitz functions on X,with norm

$$\|f\| = \max(L(f), \|f\|_\infty)$$

The resulting space is denoted $Lip(X)$.

Lipschitz algebras have been studied extensively. A reasonably rich literature exists on this subject;see ([1 − 30]). What evidence is there that Lipschitz algebras deserve a wider audience?

Perhaps the most convincing is the fact that Lipschitz algebras play a major role in Alain Connes' theory of noncommutative metric spaces(see [31]). The relation between Lipschitz algebras and noncommutative metric theory arises in two ways.

First,Lipschitz algebras are dual to metric spaces in much the same way that abelian C^*-algebras are dual to topological spaces,thus creating an analogy with noncommutative topology. The algebras $Lip(X)$ and $Lip_0(X)$ play roughly the same role here that $C(X)$ and $C_0(X)$ play in the theory of C^*-algebras. For example, the metric space X can be recovered from $Lip_0(X)$ as the set of its *weak** continuous complex homomorphisms,with metric inherited from the dual space $Lip_0(X)$([29]). Furthermore,the set of Lipschitz functions $f : X \to Y$ which preserve base point is in 1-1 correspondence with the set of *weak** continuous homomorphisms $T : Lip_0(Y) \to Lip_0(X)$,via the formula $T(g) = g \circ f$([29]). The *weak** closed ideals of $Lip_0(X)$ are in 1-1 correspondence with the closed subsets of X containing e, by pairing each ideal with its hull([29]).Finally, for every *weak** closed self-adjoint subalgebra $\mathcal{A}$ of $Lip_0(X)$,there is a metric space Y and a surjective map $f : X \to Y$ such that composition with f defines an isometric isomorphism from $Lip_0(Y)$ onto $\mathcal{A}$.

Second, unbounded derivations also arise in Connes' theory, although the approach is slightly different. By the duality between X and $Lip_0(X)$ discussed above, Lipschitz algebras play more or less the same role with respect to metric spaces as do abelian C^*- algebras with respect to topological spaces. It is therefore reasonable to expect that Lipschitz algebras will be important in the theory of noncommutative metric spaces. A standard sort of idea is that we might describe a noncommutative metric space by a norm-dense $*$-subalgebra $\mathcal{L}$ of a C^*-algebra $\mathcal{A}$. This would be analogous to the subalgebra $Lip(X) \subseteq C(X)$ for X a compact metric space, so that we would loosely think of $\mathcal{A}$ as the set of noncommutative continuous functions and $\mathcal{L}$ as the set of noncommutative Lipschitz functions. This set-up could be thought of as describing a noncommutative metric on the grounds that in the commutative case one way of defining a metric on X is by specifying the algebra of Lipschitz functions. The use of a C^*- algebra here in a sense runs counter to the commutative theory, for one does not typically bestow a metric on a topological space. Rather,one defines a metric on a set,or more generally on a measure space,and this seems to be a more basic operation than metrizing a topological space. Thus,we might instead ask for an ultraweakly dense $*$-subalgebra $\mathcal{L}$ of a von Neumann algebra $\mathcal{M}$. Of course, once $\mathcal{L}$ is given we can take $\mathcal{A}$ to be the operator norm closure of $\mathcal{L}$ in $\mathcal{M}$ and in this way describe a noncommutative topology associated to the noncommutative metric given by $\mathcal{L}$. The main question is exactly which subalgebras $\mathcal{L}$ should be thought of as describing noncommutative metrics. Obviously we would like a definition which,in the commutative case, specifies exactly the Lipschitz algebras. One natural approach would be to try to define the relevant subalgebras $\mathcal{L}$ axiomatically,based on some abstract characterization of Lipschitz algebras;however,all known characterizations invovle lattice structure and so there is no obvious way of removing the commutative hypothesis to get a noncommutative definition. The seminal idea of using derivations in this connection is implicit in the work of Connes(see[31],Chapter VI). It seems well corroborated here by our characterization of Lipschitz algebras as being precisely the domains of W^*-derivations of von Neumann algebras. This leads to the following prescription: *A noncommutative metric space is described by a von Neumann algebra $\mathcal{M}$ and a subalgebra $\mathcal{L}$ which is the domain algebra of some W^*-derivation of $\mathcal{M}$.* The analogy to noncommutative topology is summarized in the following diagram.

$$\begin{array}{ccc} metric \quad spaces & \longleftrightarrow & Lipschitz \qquad algebras \\ & & \downarrow \\ & & W^*domain \qquad algebras \end{array}$$

and

$$\begin{array}{ccc} topological \quad spaces \longleftrightarrow & abelian & C^* - algebras \\ & & \downarrow \\ & & C^* - algebras \end{array}$$

The theory of noncommutative metric spaces was initiated by Connes and has been developed in $[31-34]$. His approach is motivated by a construction involving a compact connected Riemannian spin manifold X. Namely, there is a derivation of $C(X)$ into $B(L^2(X,S))$(the bounded operators on the Hilbert space of L^2 spinors on X)given by commutation with the Dirac operator D. The domain of this derivation is $Lip(X)$. The idea at this point is that the metric on X can be recovered from $Lip(X)$,hence from the Dirac operator. Thus,the metric is encoded in the triple$(C(X), L^2(X,S), D)$. Based on this fact Connes argues that noncommutative metric spaces should be modelled by unbounded Fredholm modules,i.e. triples $(\mathcal{A}, H, D)$ consisting of a C^*-algebra $\mathcal{A}$ represented on a Hilbert space H together with an unbounded self-adjoint operator D on H. N.Weaver's approach in [35] is based on Connes',but differs from it in some important ways. First,we take as fundamental not the operator D,but rather the derivation $x \to i[D,x]$. This then leads to the more general case of abstract derivations,which can be used to model $Lip(X)$ for any metric space X(not just Riemannian manifolds). The abstract point of view also has the advantage of allowing us to distinguish the case where E is abelian.

N.Weaver's paper([35]) is the result of an attempt to understand the role of Lipschitz algebras in Alain Connes' theory of noncommutative metric spaces. It is found in [35] that the class of Lipschitz algebras is identical to the class of domains of unbounded derivations of abelian von Neumann algebras. Moreover, N.Weaver discussed in the same paper the implications of his results for the theory of noncommutative metric spaces.

Besides its close relationship to noncommutative metric spaces, Lipschitz algebras themselves have a rich and beautiful theory. For instance, N.Weaver gives the version of the Stone-Weierstrass theorem for Lipschitz algebras in [29]. Moreover, the versions of the Banach-Stone theorem for Lipschitz algebras are also given in [3] and [27].

In conclusion, Lipschitz algebras become very active and have received much attention in functional analysis and differential geometry,particularly,in noncommutative geometry. We refer any interested reader to see for instance the recent works $[35-40]$.

Based on the above consideration, we devote this report to generalizing some basic results of Lipschitz algebras to uniform spaces,i.e.,Banach spaces of uniformly continuous functions on abstract metric spaces. We have divided this report into two chapters. In Chapter 1, we study uniform spaces,i.e.,Banach spaces of uniformly continuous functions on abstract metric spaces. In Section 1.1, we give some elementary properties of scalar-valued uniformly continuous functions such as linear, order-theoretic and algebraic properites, which will be used in this and the next chapter. In Section 1.2, we define appropriate norms on the space of all bounded scalar-valued uniformly continuous functions on metric space X and the space of all scalar-valued base point-preserving uniformly continuous functions on metrically convex space X, denoted $UC(X)$ and $UC_0(X)$, respectively,and then prove they are both Banach spaces under appropriate norms. The relationships between $UC(X), UC_0(X), C_b(X), Lip(X)$ and X^* are also discussed in this section. In Section 1.3 we prove that both $Lip(X, \beta\circ\rho)$ and $Lip_0(X, \beta\circ\rho)$ are dual spaces. Moreover,we give their preduals. Section 1.4 is devoted to the extreme points of the unit ball of $UC_0(X)^*$. In this section, we first ensure the existence of extreme points under some conditions and then give the exact form that the extreme points must take. Chapter 2 is devoted to algebraic and lattice structures of $UC(X)$ and $UC_0(X)$. In Section 2.1, it is proved that $UC(X)$ and $UC_0(X)$ can be renormed to become Banach algebras, and then the Stone-Weierstrass theorem for $UC(X)$ and $UC_0(X)$ are given. In Section 2.2, it is shown that $UC_0(X)$ is a Lipschitz lattice and hence $UC_0(X)$ is lattice isomorphic to some $Lip_0(Y)$,where Y is a pointed compelte metric space with diameter at most two.

Our notation and terminology are standard as may be found in [39, 41]

Chapter 1

Uniform Spaces

Many people seem to think that a metric is nothing more than a crude version of a topology. This is not true. Certainly, every metric space has a natural topology;but just as certainly, metric spaces are more rigid than topological spaces and are really a different kind of object. For instance,the natural completeness condition in the topological setting is compactness, while the appropriate completeness condition for metric is just that,completeness. To be sure,there are many analogies between metric and topological spaces. There are metric versions of several classic topological theorems,such as the existence of embeddings in products, the Tietze extension theorem, and the Stone-Weierstrass theorem. But in other ways metric spaces are related to measure spaces,as we can see in [39]. Moreover, continuous functions are not the most natural morphisms between metric spaces. The most natural morphisms are Lipschitz functions and uniformly continuous functions. In the case of Lipschitz functions, many basic and important results have been obtained(see [39]). However, this is not the case for uniformly continuous functions. Thus spaces of scalar-valued uniformly continuous functions on abstract metric spaces are the central topic of this report. This first chapter provides the foundational material that will be needed subsequently.

Throughtout the report, the scalar field $\mathbf{F}$ could be either $\mathbf{R}$ or $\mathbf{C}$.

§1.1 Scalar-valued Uniformly Continuous Functions

Now we give the definition of uniformly continuous functions by modulus of continuity.

Definition 1.1.1 *Let X be a metric space and let $\mathbf{F}$ be scalar field. A map $f : X \to \mathbf{F}$ is said to be uniformly continuous if, for all $\epsilon > 0$, there exists $\delta > 0$ so that $|f(p) - f(q)| < \epsilon$ whenever $\rho(p, q) < \delta$; or equivalently, if there is a $t_0 > 0$ such that $\omega_f(t) < +\infty$ for $t < t_0$ and $\lim_{t \to 0^+} \omega_f(t) = 0$, where the function $\omega_f(t) = \sup\{|f(p) - f(q)| : p, q \in X, \rho(p, q) \leq t\}$(for all $t > 0$) is called to be the modulus of continuity of f.*

Remark 1.1.2 *When X is metrically convex, $\omega_f(t)$ is subadditive and nondecreasing. Thus for any scalar-valued uniformly continuous function $f : X \to \mathbf{F}$, we have $\omega_f(t) < +\infty$ for any $t > 0$.*

We begin this section with a simple observation of Proposition 1.11 (see [41]) which will be used repeatedly later and we omit the proof.

Lemma 1.1.3 *Let X be a metrically convex space. Then, for any scalar-valued uniformly continuous function $f : X \to \mathbf{F}$, we have: for every $d > 0, |f(p) - f(q)| \leq \frac{2\omega_f(d)}{d}\rho(p,q)$ whenever $\rho(p,q) \geq d$,that is, f is Lipschitz for large distance.*

The definition of modulus of continuity immediately yields the following easy result.

Proposition 1.1.4 *Let X be a metric space and let f and $f_i (i \in I)$ be functions from X to $\mathbf{F}$. Suppose $f_i \to f$ pointwise. Then,for all $t > 0$,*

$$\omega_f(t) \leq \sup_i \omega_{f_i}(t)$$

.

The next three propositions summarize the basic linear,algebraic and order-theoretic properties of scalar-valued uniformly continuous functions, which we will use in this chapter and the next chapter.

Proposition 1.1.5 *Let X be a metric space and let f, g and $f_n (n \in \mathbf{N})$ be functions from X to $\mathbf{F}$. Then,for all $t > 0$, we have:*

(a) $\omega_{af}(t) = |a|\omega_f(t)$ *for all* $a \in \mathbf{F}$;

(b) $\omega_{f+g}(t) \leq \omega_f(t) + \omega_g(t)$, *and*

(c) *if* $\sum\limits_n f_n$ *converges pointwise, then* $\omega_{\sum\limits_n f_n}(t) \leq \sum\limits_n \omega_{f_n}(t)$.

Proof. We only prove (c). Let $g_n = \sum\limits_{k=1}^{n} f_k$ and $f = \sum\limits_{k=1}^{\infty} f_k$. Then $g_n \to f$ pointwise and $\omega_{g_n}(t) \leq \sum\limits_{k=1}^{n} \omega_{f_k}(t)$. So by Proposition 1.1.4, we have

$$\omega_f(t) \leq \sup_n \omega_{g_n}(t) \leq \sum_{k=1}^{\infty} \omega_{f_k}(t)$$

This completes the proof. □

Proposition 1.1.6 *Let X be a metric space and let $f, g : X \to \mathbf{F}$ be bounded functions. Then,for all $t > 0$, we have:*

(a) $\omega_{fg}(t) \leq \|f\|_\infty \omega_g(t) + \|g\|_\infty \omega_f(t)$ *and*

(b) *if* $|f(p)| \geq \epsilon > 0$ *for all* $p \in X$, *then* $\omega_{\frac{1}{f}}(t) \leq \frac{\omega_f(t)}{\epsilon^2}$.

If X is metrically convex with finite diameter, then the product of any two scalar-valued uniformly continuous functions is again uniformly continuous.

Proof. It suffices to note that every uniformly continuous function on a finite diameter metrically convex space is bounded. Indeed, it follows from Lemma 1.1.3 that for any $p, q \in X$,

$$|f(p) - f(q)| \leq \max\{\omega_f(1), 2\omega_f(1)diam(X)\}$$

This completes the proof. $\square$

A partial order on the set of real-valued uniformly continuous functions can be defined by setting $f \leq g$ if and only if $g - f \geq 0$. Then

$$f \bigvee g = max(f, g), f \bigwedge g = min(f, g)$$

and

$$\bigvee_i f_i = \sup_i f_i, \bigwedge_i f_i = \inf_i f_i$$

Proposition 1.1.7 *Let X be a metric space and let f, g and $f_i (i \in I)$ be functions from X to $\mathbf{R}$. Then,for all $t > 0$, we have:*

$$\omega_{f \bigvee g}(t), \omega_{f \bigwedge g}(t) \leq \max\{\omega_f(t), \omega_g(t)\}$$

and if $\bigvee\limits_i f_i$ and $\bigwedge\limits_i f_i$ are finite everywhere, then

$$\omega_{\bigvee\limits_i f_i}(t), \omega_{\bigwedge\limits_i f_i}(t) \leq \sup_i \omega_{f_i}(t)$$

Proof. Let $h = f \bigvee g$ and fix $p, q \in X$ with $\rho(p, q) \leq t$. We may assume that $h(p) \geq h(q)$ and $f(p) = h(p)$. Then

$$h(p) - h(q) \leq f(p) - f(q) \leq \max\{\omega_f(t), \omega_g(t)\}$$

Thus

$$\omega_{f \bigvee g}(t) \leq \max\{\omega_f(t), \omega_g(t)\}$$

Note that $f \bigvee g = -((-f) \bigwedge (-g))$, it follows from Proposition 1.1.5 that

$$\omega_{f \bigwedge g}(t) = \omega_{-((-f) \bigvee (-g))}(t) = \omega_{(-f) \bigvee (-g)}(t) \leq \max\{\omega_f(t), \omega_g(t)\}$$

Since the joint and meet of an infinite collection are pointwise limits of the joins and meets of all finite subcollections, the rest of the proposition follows from Proposition 1.1.4. $\square$

The final result in this section shows the relationship between real- and complex-valued uniformly continuous functions.

Proposition 1.1.8 *Let X be a metric space and let $f : X \to \mathbf{C}$ be a map. Then,for all $t > 0$, we have:*

$$\max(\omega_{Ref}(t), \omega_{Imf}(t)) \leq \omega_f(t) \leq \sqrt{2}\max(\omega_{Ref}(t), \omega_{Imf}(t))$$

Proof. Note that for any $z \in \mathbf{C}$,we have:

$$\max(|Rez|, |Imz|) \leq |z| \leq \sqrt{2}\max(|Rez|, |Imz|)$$

Set $z = f(p) - f(q)$ and take the supremum over $p, q \in X$ with $\rho(p, q) \leq t$. $\square$

§1.2 Uniform Spaces

The following objects are the subject of this report.

Definition 1.2.1 *(a) Let X be a metric space. Then $UC(X)$ is the space of all bounded scalar-valued uniformly continuous functions on X, with norm $\|f\|_U = \max(\|f\|_\infty, \omega_f(1))$. We call this the uniform norm.*

(b) Let X be a pointed, metrically convex space. Then $UC_0(X)$ is the space of all scalar-valued base point-preserving uniformly continuous functions on X, with norm $\omega_f(1)$.

Remark 1.2.2 *(a) For $f \in UC(X), \omega_f(1) \leq 2\|f\|_\infty < +\infty$. Combining this with Proposition 1.1.5, we conclude that $\|\cdot\|_U$ is a norm on $UC(X)$. Thus $(UC(X), \|\cdot\|_U)$ is a normed linear space for any metric space X.*

(b) For $f \in UC_0(X)$, it follows from Lemma 1.1.3 and Proposition 1.1.5 that $(UC_0(X), \omega.(1))$ is a normed linear space for any pointed metrically convex space X.

(c) When we want to specify the scalar field, we will write e.g.$UC(X;\mathbf{R})$. Proposition 1.1.8 implies that $UC(X;\mathbf{C}) = UC(X;\mathbf{R}) + iUC(X;\mathbf{R})$,and similarly for $UC_0(X;\mathbf{C})$.

(d) Because $\omega.(1)$ of any constant function is zero, $\omega.(1)$ is only a seminorm ,not a norm, on the space of all scalar-valued uniformly continuous functions. In Definition 1.2.1, this difficulty is dealt with in two different ways. The first is to consider only bounded functions and incorporate $\|\cdot\|_\infty$ into the norm. The second is to consider only base point-preserving functions and this has precisely the effect of eliminating the constant functions. Another way of dealing with constant functions is to factor them out. That is , let X be a metrically convex space and consider the set of all real-valued uniformly continuous functions modulo the set of constant functions. It follows from Lemma 1.1.3 that $\omega_f(1) = 0$ if and only if f is constant. Thus $\omega.(1) = 0$ descends to a norm on this quotient space and it is not hard to see that this quotient space is isometrically isomorphic to $UC_0(X)$(regardless of the choice of base point). However,the problem with this procedure is that there is no good way to define products or a partial order on the quotient.

(e) $UC_0(X)$,as a Banach space, does not depend on the choice of base point. Explicitely,if e and e' are two different choices then the map $f \mapsto f - f(e')$ takes $UC_0(X)$ with base point e linearly and isometrically onto $UC_0(X)$ with base point e'. However, in general this map is not compatible with products and fails to preserve the partial order,so that both the algebraic and order structures are different for different base points. This limits the usefulness of the observation about independence of the norm.